Why Study the U.S. Conventional Algorithms?

In *Investigations,* the algorithms traditionally taught in the United States are studied by students after they have developed their own fluent methods for solving problems with whole numbers in each operation. These include the algorithms for addition, multiplication, and subtraction which involve regrouping the numbers. Historically, these algorithms were developed for doing calculations by hand with a minimum of steps and compact notation. The power of these algorithms for quick calculation lies largely in the fact that they require the user to carry out a series of mostly single-digit calculations. They were designed so that the user could rely on a small set of known number combinations and the repetition of a small sequence of steps to solve any problem. These algorithms, as human inventions, are elegant and efficient.

However, in the elementary grades, when we want students to acquire solid understanding of the base-ten number system and the meaning of arithmetic operations, these algorithms tend to obscure both the place value of digits and fundamental properties of the operations. Research and practice in the field of mathematics education have shown that there are alternative algorithms and strategies that students develop, that help them maintain a focus on understanding place value and the operations and, at the same time, are easily generalized and efficient.

Although each student may primarily use one strategy for each operation, in *Investigations,* students are expected to study more than one algorithm or strategy for each operation. Students study a variety of approaches for the following three reasons:

- Different algorithms and strategies provide access to analysis of different mathematical relationships.
- Access to different algorithms and strategies leads to flexibility in solving problems. One method may be better suited to a particular problem.
- Students learn that algorithms are "made objects" that can be compared, analyzed, and critiqued according to a number of criteria.

As the NCTM's *Principles and Standards for School Mathematics* (2000) states:

> Many students enter Grade 3 with methods for adding and subtracting numbers. In Grades 3–5, they should extend these methods to adding and subtracting larger numbers, and learn to record their work systematically and clearly. Having access to more than one method for each operation allows students to choose an approach that best fits the numbers in a particular problem. (page 155)

In students' study of calculation methods for each operation, they first build strategies that they are comfortable with, that make sense to them, that they can use fluently, and that can gradually be applied to harder problems. At a later time they study some of the strategies they are less comfortable with in order to learn about the underlying mathematical relationships. This later period includes a study of conventional algorithms that are commonly used in the students' communities. This study of conventional algorithms has both a mathematical and a social purpose.

Students with good understanding of an operation—what it is used for, what its properties are, how to solve a problem that requires that operation efficiently, how it is related to other operations, and how the base-ten number system is used in that operation—can use a study of *any* algorithm that has been invented for that operation as an opportunity to delve further into the operation itself. Studying how and why an unfamiliar algorithm works is a challenge to think through what we know about an operation. It requires pulling apart an algorithm, bringing meaning to shortcut notations, and finding parts of the algorithm that are similar to parts of more familiar algorithms.

For example, one of the authors once noticed an older relative using the following algorithm to solve subtraction problems:

$$\begin{array}{r} 7\,{}^{1}5 \\ -\,{}^{4}39 \\ \hline 36 \end{array}$$

This woman had been educated in the United States in the early part of the twentieth century when this algorithm had been "standard" in some places in the country. By thinking through what this shortcut notation means, we can see that adding 10 to each number in the original problem, (70 + 5) − (30 + 9), gives us an equivalent problem, (70 + 15) − (40 + 9), that can be solved by subtracting each place. The underlying principle is that changing the two numbers in a subtraction problem by adding (or subtracting) the same amount results in a problem with the same answer as the original problem. Thinking through *why* an algorithm works brings us back to fundamental ideas about the operations, ideas on which these algorithms are based.

Another mathematical reason for studying these algorithms is that they have been used and found useful by many people. Too often in the past, these algorithms were taught and learned without meaning. And, too often, these algorithms were seen as the central teaching tool for learning about an operation: learning addition was defined as learning the steps of the "carrying" algorithm. However, whereas the "carrying" algorithm may have held an inappropriately central place in our teaching strategies at one time, it is a perfectly good algorithm that can be used by those who find it useful. Competent adults often use different algorithms for different contexts, use a mixture of algorithms, or use one algorithm or strategy to check another. For example, one of the authors has a particular algorithm for subtraction that she uses only in her checkbook (it is neither the standard borrowing algorithm nor any of those used in this curriculum)—it is one that she has shaped to fit her particular needs in that context. Therefore, a second reason for studying the carrying and borrowing algorithms is to provide students exposure to these algorithms and their underlying meaning. Those who find them sensible and useful may choose to adopt them for their own uses in life.

The third reason for studying conventional algorithms is that they are a part of the social knowledge in students' communities. Adults in students' lives may use these algorithms, and they need not be a mystery to students. Because a variety of algorithms have been taught in different countries and at different times in the U.S. (as, for example, the subtraction algorithm shown above), we recommend that you have students bring in algorithms used by adults in their families. You may find that there is more than one algorithm commonly used in the students' community for a particular operation.

The following are two primary goals for the study of numbers and operations in the elementary grades:

1. Understanding the meaning and properties of the operations

2. Attaining computational fluency with whole numbers

These goals underlie the choices we make in the study of algorithms and strategies. As stated in NCTM's *Principles and Standards:*

> Students should come to view algorithms as tools for solving problems rather than as the goal of mathematics study. As students develop computational algorithms, teachers should evaluate their work, help them recognize efficient algorithms, and provide sufficient and appropriate practice so that they become fluent and flexible in computing. (page 144)

Teacher Note

Computational Algorithms and Methods

In the elementary grades, a central part of students' work is learning about addition, subtraction, multiplication, and division and becoming fluent and flexible in solving whole number computation problems. In the *Investigations* curriculum, students use methods and algorithms in which they can see clearly the steps of their solution and focus on the mathematical sense of what they are doing. They use and compare several different methods to deepen their understanding of the properties of the operations and to develop flexibility in solving problems. They practice methods for each operation so that they can use them efficiently to solve problems.

What Is an Algorithm?

An algorithm is a series of well-defined steps used to solve a certain class of problem (for example, all addition problems). Often, the sequence of steps is repeated with successive parts of the problem. For example, here is an example of an addition algorithm:

$249 + 674$

$200 + 600 = 800$

$40 + 70 = 110$

$9 + 4 = 13$

$800 + 110 + 13 = 923$

Written instructions for this algorithm might begin as follows:

1. Find the left-most place represented in the addends and add all the amounts in that place.
2. Move one place to the right and add all the amounts in that place in all the addends.
3. Repeat step 2 until all parts of all addends have been added.
4. Add the sums of each place.

To specify these instructions, as if we were going to teach them to a computer, we would have more work to do to make them even more specific and precise. For example, how is step 4 carried out? Should each place be added separately again and then combined? In practice, when students and adults use this algorithm, the partial sums that must be added in step 4 are generally easy enough to add mentally, as they are in this problem, although occasionally one might again break up some of the numbers.

Algorithms like this one, once understood and practiced, are general methods that can be used for a whole class of problems. The adding by place algorithm, for example, can be generalized for use with any addition problem. As students' knowledge of the number system expands, they learn to apply this algorithm to, for example, larger numbers or to decimals. Students also learn how to use clear and concise notation, to carry out some steps mentally, and to record those intermediate steps needed so that they can keep track of the solution process.

Nonalgorithmic Methods for Computing with Whole Numbers

Students also learn methods for computing with whole numbers that are not algorithmic—that is, one cannot completely specify the steps for carrying them out, and they do not generally involve a repetition of steps. However, these methods are studied because they are useful for solving certain problems. In thinking through why and how they work, students also deepen their understanding of the properties of the various operations. This work provides opportunities for students to articulate generalizations about the operations and to represent and justify them.

For example, here is one method a third grader might use to solve this problem:

$\$7.46 + \$3.28 = \$7.50 + \$3.24 = \$10.74$

The student changed the addition expression to an equivalent expression with numbers that made it easier to find the sum mentally. First graders often use this idea as they learn some of their addition combinations, transforming a combination they are learning into an equivalent combination they already know: $7 + 5 = 6 + 6 = 12$.

When students try to use the same method to make a subtraction problem easier to solve, they find that they must modify their method to create an equivalent problem. Instead of adding an amount to one number and subtracting it from the other, as in addition, they must add the same amount to (or subtract it from) each number:

$182 - 69 = 183 - 70 = 113$

Throughout the *Investigations* curriculum, methods like these are introduced and studied to deepen students' understanding of how these operations work and to engage them in proving their ideas using representations of the operations.

Because the ways in which a problem might be changed to make an equivalent problem that is easier to solve can vary (although it might be possible to precisely specify a particular variant of one of these methods), these methods are not algorithms. Students do not generally use such methods to solve a whole class of problems (e.g., any addition problem); rather, students who are flexible in their understanding of numbers and operations use finding equivalent expressions as one possible method and notice when a problem lends itself to solving in this way.

Learning Algorithms Across the Grades

In *Investigations,* students develop, use, and compare algorithms and other methods. These are not "invented" but are constructed with teacher support, as students' understanding of the operations and the base-ten number system grow (see the Teacher Note, Computational Fluency and Place Value). Because the algorithms that students learn are so grounded in knowledge of the operation and the number system, most of them arise naturally as students progress from single-digit to multidigit problems. For example, the adding by place addition algorithm shown earlier naturally grows out of what students are learning about how a number such as 24 is composed of 2 tens and 4 ones. It is part of the teacher's role to make these methods explicit, help students understand and practice them, and support students to gradually use more efficient methods. For example, a second grader who is adding on one number in parts might solve $49 + 34$ by adding on 10, then another 10, then another 10, then 4 to 49 ($49 + 10 + 10 + 10 + 4$). By having this student compare solutions with another student's whose first step is $49 + 30$, the teacher helps the first student analyze what is the same and different about their solutions and opens up the possibility for the first student of a more efficient method—adding on a multiple of 10 all at once rather than breaking it into 10s.

The algorithms and other methods that students learn about and use in *Investigations* for multidigit problems are characterized by their *transparency.* Transparent algorithms

- make the properties of the operations visible.
- show the place value of the numbers in the problem.
- make clear how a problem is broken into subproblems and how the results of these subproblems are recombined.

These characteristics are critical for students while they are learning the meaning of the operations and are building their understanding of the base-ten system. Here is an example of a transparent multiplication algorithm that might be used by a fourth grader:

$$\begin{array}{r} 34 \\ \underline{\times\ 78} \\ 2100 \\ 280 \\ 240 \\ \underline{32} \end{array}$$

$$2{,}000 + 500 + 150 + 2 = 2{,}652$$

In this algorithm, students record all numbers fully, showing the place value of all the digits. Because the result of each multiplication is shown, the application of the distributive property is kept track of clearly.

There is a misperception that many different algorithms might arise in a single classroom and that this multitude of algorithms will be confusing. In fact, there are only a few basic algorithms and methods for each operation that arise from students' work and that are emphasized in the curriculum. Each is tied closely to how students solve problems and to the basic characteristics and properties of the operation. Teacher Notes throughout the curriculum provide more detail about these methods.

Students can and do develop efficiency and fluency with these more transparent algorithms. As they do, they do some steps mentally and may no longer need to write out every step to keep track of their work. For example, in using the adding by place algorithm to add 249 + 674, a competent user might simply jot down 800, 110, 13, and then add those partial sums mentally and record the answer. There may be times when you require students to write out their complete solution method so that you can see how they are solving problems, but for everyday use, efficient users of such algorithms will record only the steps they need.

These algorithms and methods are studied, compared, and analyzed for different reasons. All of them are transparent, preserve place value, and make visible important properties such as distributivity. Some can be practiced and provide general, efficient methods. Others are useful only for particular problems but are studied because of what they illuminate about the operations.

Studying the U.S. Standard Algorithms

The U.S. standard algorithms for addition, subtraction, and multiplication are also explicitly studied in *Investigations* but only after students are fully grounded in understanding the operation and using transparent algorithms for multidigit computation. These algorithms were developed for efficiency and compactness for handwritten computation. When these algorithms are used as a primary teaching tool, their very compactness, which can be an advantage for experienced users, becomes a disadvantage for young learners because they obscure the place value of the numbers and the properties of the operation.

Some students do use the standard algorithms with understanding. As these algorithms come up in class, they should be incorporated into the list of class strategies. Teachers should make sure that students who use them understand what the shortcut notation represents and that they can explain why these algorithms make sense. They should also know and understand other methods. In Grade 4, students revisit the U.S. standard addition algorithm formally, analyze how and why it works, and compare it to other algorithms they are using. In Grade 5, students revisit the U.S. standard subtraction and multiplication algorithms in the same way. Division methods studied in this curriculum focus on the inverse relationship between multiplication and division.

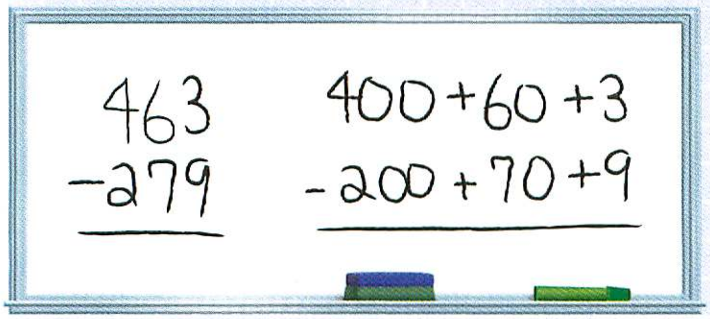
Working with the U.S. Algorithm

During the past few days, students in this Grade 5 class have been sharing, exploring, and using various subtraction strategies. One of the strategies that several students use is the U.S. algorithm. In this session, students study this strategy, trying to understand its notation. The class has already looked at one example, 674 − 328, in which one ten is "borrowed." Now they look at a more difficult problem.

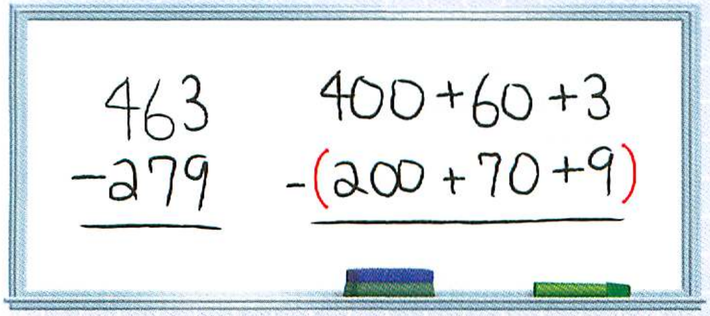
$$\begin{array}{r} 463 \\ -\ 279 \\ \hline \end{array}$$

Teacher: We're going to solve this next problem by looking carefully at what happens in each place. Can someone come up and rewrite the numbers by place value?

Martin comes up to the board and rewrites the problem:



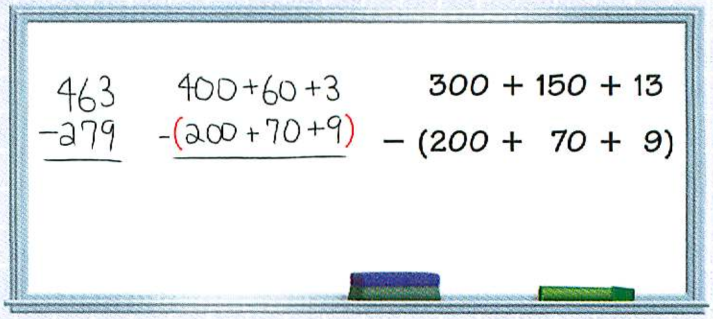
The teacher adds parentheses to what Martin wrote on the board:



Teacher: I wrote the parentheses to show that we are subtracting all the parts of the bottom number. Let's try to solve this problem, the way they do in the U.S. algorithm. Martin, what's the first thing you're going to do?

Martin: I start in the ones place. If I subtract 9 from 3, I get a negative number, so I have to trade for a 10. Wait a minute, I have to trade for a 100! You need to take 100 from the 400 and add that 100 to the 60. Then take a 10 from the 160 and add it to the 3. That will make the 160 a 150.

The teacher adds to what Martin wrote on the board:



Teacher: Is that what you mean?

Martin: Yes.

Teacher: Let's slow down a minute because Martin combined two steps. Martin, you started to trade for a 10 and then said you had to trade for a 100. Why?

Martin: When I looked at the 60, I realized that I couldn't subtract the tens either without getting a negative number, so I just got a 100 and gave it to the 60.

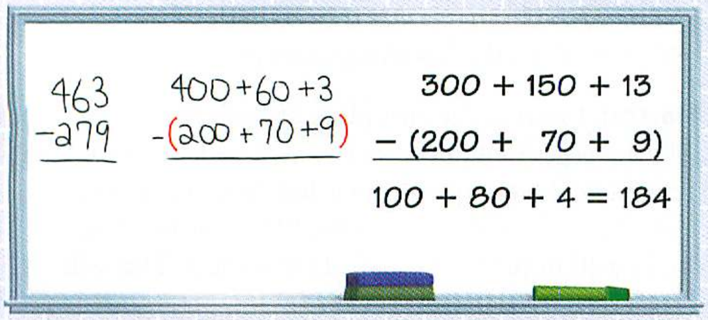
Teacher: Good. So what should the top numbers add up to?

Lourdes: 463.

Teacher: Who can finish the problem?

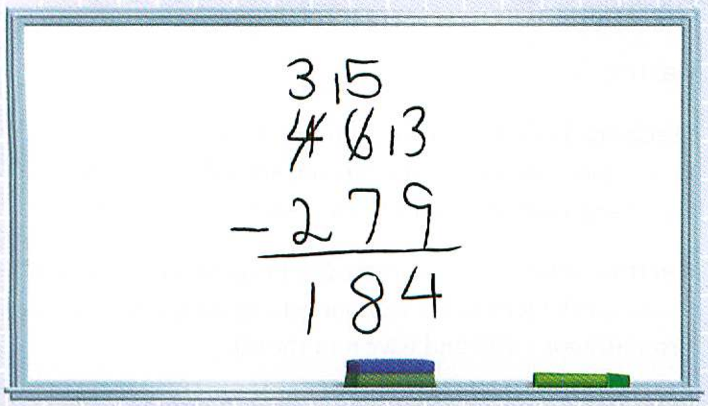
Hana: 300 − 200 is 100, 150 − 70 is 80, and 13 − 9 is 4. I add them all up and it's . . . 184.

The teacher adds Hana's responses to the problems on the board:



Teacher: What we did is separate the numbers to show how the algorithm works, but when people use the actual algorithm, they don't write out all the numbers. Let's see what that looks like. Tavon, you use this strategy often. Can you show us, using this problem?

Tavon: 3 take away 9 and I can't do that, I mean it's a negative, so I cross out the 6 tens and make it a 5 and then make that a 13. 13 minus 9 is 4. Then 5 − 7, I can't do that, so I cross out the 4 and make it a 3 and 15 minus 7 is 8.



Teacher: Hold on a second. Let's go over that. Where did you get that little 1 you put next to the 3 and what does that mean?

Tavon: When I crossed out the 6 and made it 5, I was taking 1 ten. Then I added 1, I mean 10, to the 3, so I had 13.

Teacher: So you had 63, but you made it into 50 + 13, which is still 63. Then what?

Tavon: Then 5 minus 7, I can't do that either, so I cross out the 4 and make it a 3 and 15 minus 7 is 8.

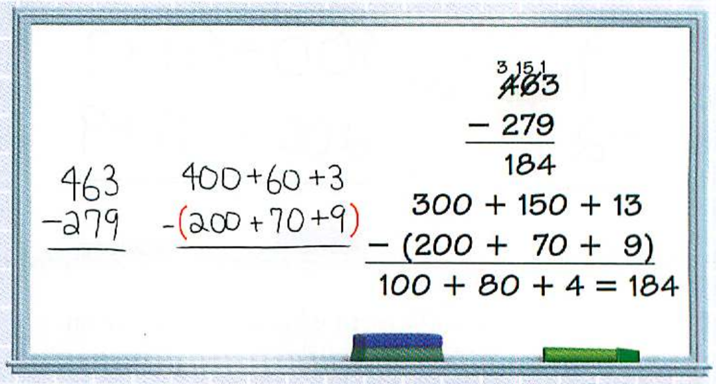
Teacher: 15 is really 15 what?

Tavon: 15 tens.

Teacher: So you're subtracting 7 tens from 15 tens. Does everybody get that?

Tavon: All I have left is 3 minus 2 equals 1, I mean 100, so the answer's the same.

Teacher: So if we look at what Tavon did and what Martin did, we should see some similarities.



Teacher: Everyone is going to try a few problems with this algorithm and try to understand the notation. If you don't like it, that's fine. But just like you are learning about other strategies you may not use, learning about this one helps you continue to learn more about different ways to think about subtraction.

The teacher encourages students to practice different strategies and notation and also stresses that sense-making is vital.

Comparing Addition Notation

564 + 278 =

Jake and Anna solved this problem by adding by place. Their solutions are similar, but they recorded their work differently.

Jake's solution

$$\begin{array}{r} 564 \\ +\ 278 \\ \hline 700 \\ 130 \\ 12 \\ \hline \mathbf{842} \end{array}$$

Anna's solution (U.S. Algorithm)

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 564 \\ +\ 278 \\ \hline \mathbf{842} \end{array}$$

The students in Jake and Anna's class compared the notation used in these two solutions. Here are some of the things they noticed:

Both solutions involve breaking numbers apart by place.
Jake added the hundreds first, then the tens, and then the ones.
Anna added the ones first, then the tens, and then the hundreds.

The little numbers in the U.S. algorithm stand for 10s and 100s.

The strategies are mostly the same, but the U.S. algorithm notation combines steps.

The last step in Jake's solution is the same as the first step in Anna's solution: 4 + 8 = 12.

In the U.S. algorithm you "carry" 10 ones to the tens place, and you "carry" 10 tens to the hundreds place.

Subtraction Strategies (page 4 of 4)

$$\begin{array}{r} 3{,}726 \\ -\ 1{,}584 \\ \hline \end{array}$$

Subtracting by Place

Yumiko subtracted by place. She combined positive and negative results to find her answer.

Yumiko's solution

$$\begin{array}{r} 3{,}726 \\ -\ 1{,}584 \\ \hline 2 \\ -\ 60 \\ 200 \\ 2{,}000 \\ \hline \mathbf{2{,}142} \end{array}$$

This notation shows each step in Yumiko's solution.

$$\begin{array}{rrrrrrrrr} & 3{,}000 & + & 700 & + & 20 & + & 6 & \\ - & (1{,}000 & + & 500 & + & 80 & + & 4) & \\ \hline & 2{,}000 & + & 200 & + & -60 & + & 2 & = \mathbf{2{,}142} \end{array}$$

Avery subtracted by place, using the U.S. algorithm.

Avery's solution

$$\begin{array}{r} {}^{6}\ \ \\ 3{,}\not{7}{}^{1}26 \\ -\ 1{,}584 \\ \hline \mathbf{2{,}142} \end{array}$$

This notation shows each step in Avery's solution.

$$\begin{array}{rrrrrrrrr} & & & 600 & & 100 & & & \\ & 3{,}000 & + & \not{700} & + & 20 & + & 6 & \\ - & (1{,}000 & + & 500 & + & 80 & + & 4) & \\ \hline & 2{,}000 & + & 100 & + & 40 & + & 2 & = \mathbf{2{,}142} \end{array}$$

How would you solve the problem 3,726 – 1,584?

Comparing Multiplication Algorithms

Math Words
- **algorithm**

Some fifth graders compared these two algorithms.

An algorithm is a step-by-step procedure to solve a certain kind of problem.

Partial Products

	278
	x 35
(5 x 8) →	40
(5 x 70) →	350
(5 x 200) →	1,000
(30 x 8) →	240
(30 x 70) →	2,100
(30 x 200) →	6,000
	9,730

U.S. Algorithm

2 2	
3 4	
278	
x 35	
1,390	← (5 x 278)
8,340	← (30 x 278)
9,730	

Here are some of the things the students noticed.

- *Both solutions involve breaking apart numbers.*
- *The first three numbers in the partial products algorithm are combined in the first number in the solution using the U.S. algorithm.*
- *The algorithms are mostly the same, but the U.S. algorithm notation combines steps (40 + 350 + 1,000 = 1,390).*
- *The little numbers in the U.S. algorithm stand for tens and hundreds. The 4 and the 2 above the 7 are really 40 and 20.*